Error analysis for a class of numerical differentiator: application to state observation

Da-Yan Liu, Olivier Gibaru, Wilfrid Perruquetti

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Abstract— **In this note, firstly a modified numerical differentiation scheme is presented. The obtained scheme is rooted in [22], [23] and uses the same algebraic approach based on operational calculus. Secondly an analysis of the error due to a corrupting noise in this estimation is conducted and some upper-bounds are given on this error. Lastly a convincing simulation example gives an estimation of the state variable of a nonlinear system where the measured output is noisy.**

I. Introduction

Since the seminal paper by Diop & Fliess ([12] see also [2]), observation theory and identifiability are closely linked to numerical differentiation scheme. Indeed, a non-linear system is observable if, and only if, any state variable is a differential function of the control and output variables, i.e., a function of those variables and their derivatives up to some finite order.

Recent algebraic parametric estimation technics for linear systems [10], [15] have been extended to various problems in signal processing (see, e.g., [11], [21], [24], [25], [29], [30], [31]). Let us emphasize that those methods, which are algebraic and non-asymptotic, exhibit good robustness properties with respect to corrupting noises, without the need of knowing their statistical properties[1]. It appears that these technics can also be used to derive numerical differentiation algorithms exhibiting similar properties see ([22], [23]). Such technics being used in [13], [14], [2] for state estimation.

Numerical differentiation is concerned with the estimation of derivatives of noisy time signals. This problem has attracted a lot of attention from different point of view

- observer design in the control literature (see [4], [5], [16], [17], [20], [28])
- digital filter in signal processing (see [1], [3], [6], [26], [27])

for on-line application which are alternative solutions to the very classical one, based on least-squares polynomial fitting or (spline) interpolation mostly used in off-line applications ([7], [18]).

D. Liu, O. Gibaru, W. Perruquetti are with Équipe Projet ALIEN, INRIA Lille-Nord Europe, Parc Scientifique de la Haute Borne 40, avenue Halley Bât.A, Park Plaza, 59650 Villeneuve d'Ascq, France. O. Gibaru is with Arts et Metiers ParisTech centre de Lille, L2MA, 8 Boulevard Louis XIV, 59046 Lille Cedex, France, W. Perruquetti is with LAGIS (CNRS, UMR 8146), École Centrale de Lille, BP 48, Cité Scientifique, 59650 Villeneuve d'Ascq, France. E-mail: dayan.liu@inria.fr, olivier.gibaru@ensam.eu, wilfrid.perruquetti@ec-lille.fr

[1]See [8], [9] for more theoretical details. The robustness properties have already been confirmed by numerous computer simulations and several laboratory experiments.

In recent papers [22], [23], numerical differentiation is revised using an algebraic framework. Nevertheless, a weakness of the above method was a lack of any error analysis, when they were implemented in practice.

The aim of this paper is twofold: to extend the numerical differentiation scheme presented in [22], [23] and to derive an error analysis when the statistical properties of the corrupting noise is known. The paper is organized as follows: Section II presents a causal estimator and the affine causal estimator extending the one obtained in [22], [23], then Section III provides an analysis of error due to a corrupting noise in these estimators and gives some upper-bounds on this error, especially the gaussian noise is considered, lastly Section IV provides a convincing simulation for the estimation of the state variable of a nonlinear system where the measured output is noisy.

II. Causal numerical differentiator

Let $y(t) = x(t) + \varpi(t)$ be a noisy observation on a finite open time interval $I \subset \mathbb{R}^+$ of a real valued smooth signal x, the successive derivatives of which we want to estimate, and $\varpi(t)$ denotes a noise. Let n be a positive integer, we are going to estimate the n^{th} order derivative of x. Let us ignore the noise $\varpi(t)$ for the moment. Assume that $x(t)$ is an analytic function on I, and for $t_0 \in I$, let us introduce

$$X(t) = \sum_{i=0}^{L} a_i \, x(t_0 + \beta_i t), \tag{1}$$

where $L \in \mathbb{N}$, $a_i \in \mathbb{R}^*$ and $\beta_i \in \mathbb{R}^*$ with $\beta_0 < \beta_1 < \cdots < \beta_L < 0$. This analytic function $X(t)$, will enable us to perform any derivatives estimations of x at point t_0 in only one general framework. Consequently, $X(t)$ is also an analytic function on $D := \{t \in \mathbb{R}^+; \forall i \in \{1, ..., L\}, \, t_0 + \beta_i t \in I\}$. The Taylor series expansion of X at t_0 is given by

$$\forall t \in D, \; X(t) = \sum_{i=0}^{L} a_i \sum_{j=0}^{+\infty} \frac{(\beta_i t)^j}{j!} x^{(j)}(t_0). \tag{2}$$

For $N \geq n$, we consider the following truncated Taylor expansion of X on $\mathbb{R}^+$: $\forall t \in \mathbb{R}^+$,

$$\begin{aligned} X_N(t) &= \sum_{i=0}^{L} a_i \sum_{j=0}^{N} \frac{(\beta_i t)^j}{j!} x^{(j)}(t_0) \\ &= \sum_{j=0}^{N} \left(\sum_{i=0}^{L} a_i \beta_i^j \right) \frac{t^j}{j!} x^{(j)}(t_0). \end{aligned} \tag{3}$$

Since X_N is a polynomial defined on $\mathbb{R}^+$ of degree N, we can apply the Laplace transform to (3):

$$\hat{X}_N(s) = \sum_{j=0}^{N} c_j s^{-(j+1)} x^{(j)}(t_0), \tag{4}$$

where $\hat{X}_N(s)$ is the Laplace transform of $X_N(t)$, $c_j = \sum_{i=0}^{L} a_i \beta_i^j$, and c_n is supposed to be different from zero. In all the sequel, the Laplace transform of a signal $u(t)$ will be denoted as $\hat{u}(s)$. To simplify the notation, the argument s will be dropped and we write it as $\hat{u}$ for short.

The basic step towards the estimation of $x^{(n)}(t)$, for $t \geq 0$ is the estimation of the coefficient $x^{(n)}(t_0)$ from the observation $y(t)$. All the terms $c_j s^{-(j+1)} x^{(j)}(t_0)$ in (4) with $j \neq n$, are consequently considered as undesired terms which we proceed to annihilate. For this, it suffices to find a linear differential operator of the form

$$\Pi = \sum_{\text{finite}} \left(\prod_{\text{finite}} \varrho_l(s) \frac{d^l}{ds^l} \right), \quad \varrho_l(s) \in \mathbb{C}(s), \tag{5}$$

such that

$$\Pi\left(\hat{X}_N(s)\right) = \varrho(s)\, x^{(n)}(t_0),$$

for some rational function $\varrho(s) \in \mathbb{C}(s)$. Such a linear differential operator is subsequently called an *annihilator* for $x^{(n)}(t_0)$. When the sum in (5) is reduced to a single term, we obtain a particular case of such linear differential operator which is a finite product.

A. Causal estimators of the derivatives of noisy signal

We investigate in this section some detailed properties and performances of a class of pointwise derivative estimators. These estimators will be derived from a particular family of annihilators. Let us recall the following annihilator used in [23] with $\nu = N+1+\mu, \mu \geq 0,\ k \geq 0$,

$$\Pi_{k,\mu}^{N,n} = \frac{1}{s^\nu} \frac{d^{n+k}}{ds^{n+k}} \frac{1}{s} \frac{d^{N-n}}{ds^{N-n}} s^{N+1}. \tag{6}$$

As we can see in (4), we have a polynomial of degree N by multiplying by s^{N+1}. Then we annihilate the terms of degree lower than $(N-n)$ by applying $(N-n)$ times derivations w.r.t s. For preserving the term in $x^{(n)}(t_0)$, we multiply the remaining polynomial by $1/s$. In order to annihilate the other terms including $x^{(i)}(t_0)$ with $i \neq n$, we apply more than n times derivations w.r.t s. Finally, we multiply by $1/s^\nu$ to obtain an integral estimator.

Let us estimate the n^{th} order derivative of x with $0 \leq n \leq N$. To do that, we will use (4) by taking[2] $X(t) = x(t_0 + \beta t)$ with $\beta < 0$, and to which we will apply the annihilator $\Pi_{k,\mu}^{N,n}$. Thus, we will have a family of causal estimators.

Proposition 1: An estimate of the derivative $x^{(n)}$ for any point $t_0 \in I$ is given by:

$$\tilde{x}_{t_0}^{(n)}(k,\mu,\beta T,N) = \frac{1}{(\beta T)^n} a_{k,\mu,n,N} \sum_{i=0}^{N-n} b_{n,N,i}\ K_i, \tag{7}$$

with

$$a_{k,\mu,n,N} = (-1)^{n+k} \frac{(\nu+n+k)!}{(n+k)!(N-n)!},$$

$$b_{n,N,i} = \binom{N-n}{i} \frac{(N+1)!}{(n+i+1)!},$$

$$c_{k,\mu,n,N,j} = \frac{(-1)^{i+j}}{(\nu+k-i-j-1)!} \binom{n+k}{j} \frac{(n+i)!}{(i+j-k)!},$$

$$K_i = \sum_{j=max(0,k-i)}^{n+k} c_{k,\mu,n,N,j} \int_0^1 p_{k,\mu,N,i,j}(\tau)\, y(\beta T\tau + t_0) d\tau,$$

$$p_{k,\mu,N,i,j}(\tau) = (1-\tau)^{\nu+k-i-j-1} \tau^{i+j}.$$

The causal estimator $\tilde{x}_{t_0}^{(n)}(k,\mu,\beta T,N)$ $(\beta < 0)$ is obtained by using the integral window $[t_0 + \beta T, t_0] \subset I$ with $k \geq 0$, $\mu \geq 0$, $T > 0$ and $\nu = N+1+\mu$.

Proof. Let $X(t) = x(t_0 + \beta t)$ with $\beta < 0$ and $t > 0$, then (4) becomes

$$\hat{X}_N(s) = \sum_{i=0}^{N} \beta^i s^{-(i+1)} x^{(i)}(t_0), \tag{8}$$

where $\hat{X}_N(s)$ is the Laplace transform of $X_N(t)$. We proceed to annihilate the terms including $x^{(i)}(t_0)(i \neq n)$ in the right hand side in equation (8) by multiplying by the annihilator $\Pi_{k,\mu}^{N,n}$ defined in (6). It reads as

$$\begin{aligned}\Pi_{k,\mu}^{N,n}\left(\hat{X}_N\right) &= \frac{1}{s^\nu} \frac{d^{n+k}}{ds^{n+k}} \sum_{i=0}^{n} \beta^i \frac{(N-i)!}{(n-i)!} s^{n-i-1} x^{(i)}(t_0) \\ &= \frac{\beta^n (N-n)!(-1)^{n+k}(n+k)!}{s^{1+n+k+\nu}} x^{(n)}(t_0).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Pi_{k,\mu}^{N,n}\left(\hat{X}_N\right) &= \frac{1}{s^\nu} \frac{d^{n+k}}{ds^{n+k}} \sum_{i=0}^{N-n} \frac{\binom{N-n}{i}(N+1)!}{(n+i+1)!} s^{n+i} (\hat{X}_N)^{(i)} \\ &= \sum_{i=0}^{N-n} \frac{\binom{N-n}{i}(N+1)!}{(n+i+1)!} \bar{F}_i.\end{aligned}$$

with

$$\bar{F}_i = \sum_{j=\max(0,k-i)}^{n+k} \frac{\binom{n+k}{j}(n+i)!}{(i+j-k)!} \frac{(\hat{X}_N)^{(i+j)}}{s^{\nu+k-i-j}}.$$

So, we have

$$\frac{x^{(n)}(t_0)}{s^{\nu+n+k+1}} = \frac{(-1)^{n+k}}{\beta^n (n+k)!(N-n)!} \sum_{i=0}^{N-n} \frac{\binom{N-n}{i}(N+1)!}{(n+i+1)!} \bar{F}_i. \tag{9}$$

As $\nu + k - i - j \geq 1$ we can express (9) back into the time domain by using the classical rules of operational calculus and the Cauchy formula for repeated integrals:

$$x^{(n)}(t_0) = \frac{(-1)^{n+k}}{\beta^n T^{\nu+n+k}} \frac{(\nu+n+k)!}{(n+k)!(N-n)!} \sum_{i=0}^{N-n} \sum_{j=\max(0,k-i)}^{n+k} \frac{\binom{N-n}{i}(N+1)!}{(n+i+1)!} A_{ij}, \tag{10}$$

[2] extension to the general case being computationally complex.

with

$$A_{ij} = \frac{(-1)^{i+j}}{(\nu+k-i-j-1)!}\binom{n+k}{j}\frac{(n+i)!}{(i+j-k)!} \int_0^T (T-\tau)^{\nu+k-i-j-1}\tau^{i+j}\, x_N(\beta\tau+t_0)d\tau.$$

We then replace $x_N(\beta\tau+t_0)$ in (10) by the noisy observed signal $y(\beta\tau+t_0)$ in order to obtain a family of strictly proper estimators, which are parameterized by k, μ, T and N. The proof can be achieved by applying the following change of variable: $\tau \to T\tau$. ■

Remark 1: Let us look at (7) which provides $\tilde{x}_{t_0}^{(n)}(k,\mu,\beta T,N)$. Since $\beta < 0$, we can take $X(t) = x(t_0 - t)$ to obtain an estimator $\tilde{x}_{t_0}^{(n)}(k,\mu,-\bar{T},N)$ which is equal to $\tilde{x}_{t_0}^{(n)}(k,\mu,\beta T,N)$ with $\bar{T} = \beta T$. So we can denote the causal estimator as $\tilde{x}_{t_0}^{(n)}(k,\mu,-T,N)$.

If $N = n$, we will use the simplified notation $\tilde{x}_{t_0}^{(n)}(k,\mu,\beta T)$ and call it a minimal causal estimator (estimate of the n^{th} order derivative based on an n^{th} order truncated Taylor expansion).

By writing $R_N(-T\tau+t_0) = x(-T\tau+t_0) - x_N(-T\tau+t_0)$ and $y(-T\tau+t_0) = x_N(-T\tau+t_0) + R_N(-T\tau+t_0) + \varpi(-T\tau+t_0)$, the estimation of $x^{(n)}(t_0)$ defined in Proposition 1 is corrupted by two sources of error: the bias term error which comes from the truncation of the Taylor series expansion of x and the noise error contribution.

B. *Affine causal estimator*

It was shown in [22], [23] that the estimator defined in Proposition 1 (with $L = 0$ and $\beta = -1$) can be written as an affine combination of some minimal causal estimators. This affine estimator corresponds to a point in the $\mathbb{Q}$-affine hull of the set

$$S_{k,\mu,T,q} = \left\{\tilde{x}_{t_0}^{(n)}(k+q,\mu,-T),\dots,\tilde{x}_{t_0}^{(n)}(k,\mu+q,-T)\right\} \tag{11}$$

where $q = N - n$. A new affine estimator was introduced in [23], which corresponds to a point in the $\mathbb{R}$-affine hull of the set (11). Moreover, as we shall shortly see in the following proposition, the Jacobi orthogonal polynomials are inherently connected with this estimator.

Definition 1: Let n, N, k, $\mu \in \mathbb{N}$ and a real $\xi \in [0,1]$, then we define an affine causal estimator of the n^{th} order derivative of x at t_0 by

$$\tilde{x}_{t_{0-}}^{(n)}(k,\mu,T,N,\xi) := \sum_{l=0}^{q} \lambda_l(\xi)\, \tilde{x}_{t_{0-}}^{(n)}(k_l,\mu_l,-T), \tag{12}$$

where $[t_0 - T, t_0] \in I$, $\lambda_l(\xi) \in \mathbb{R}$ and $(k_l,\mu_l) = (k+q+l, \mu+l)$.

Proposition 2: [23] Let $\tilde{x}_{t_0}^{(n)}(k,\mu,-T,N,\xi)$ be an affine causal estimator. Assume that $q \le k+n$ with $q = N-n$, then for any $\xi \in [0,1]$, there exists a unique set of real coordinates $\lambda_l(\xi) \in \mathbb{R}$, for $l = 0,\dots,q$, such that

$$\tilde{x}_{t_0}^{(n)}(k,\mu,-T,N,\xi) = x_{LS,q}^{(n)}(-T\xi) + e_\varpi(t_0;k,\mu,-T,n,N,\xi),$$

where

$$x_{LS,q}^{(n)}(-T\xi) := \sum_{i=0}^{q} \frac{\langle P_i^{k,\mu}(\tau), x^{(n)}(-T\tau+t_0)\rangle}{\|P_i^{k,\mu}\|^2} P_i^{k,\mu}(\xi),$$

$$e_\varpi(t_0;k,\mu,-T,n,N,\xi) = \sum_{l=0}^{q} \lambda_l(\xi)\, e_\varpi(t_0;k_l,\mu_l,-T,n).$$

The $P_i^{k,\mu}(\cdot)$ denotes the Jacobi polynomial and $x_{LS,q}^{(n)}(-T\xi)$ denote the least-squares q^{th} order polynomial approximation of $x^{(n)}(\cdot)$ in the interval $[-T+t_0, t_0]$.

Proof. See [23] for the original proof and the calculation of $\lambda_l(\xi)$ for $l = 0,\cdots,q$. ■

It was shown in [23] that when $\xi = 0$, the estimator $\tilde{x}_{t_0}^{(n)}(k,\mu,-T,N,0)$ is equal to $\tilde{x}_{t_0}^{(n)}(k,\mu,-T,N)$ defined in Proposition 1. So Proposition 2 gives a general causal estimator. It can be written as

$$\tilde{x}_{t_0}^{(n)}(k,\mu,-T,N,\xi) = \int_0^1 p_{k,\mu,-T,n,N,\xi}(\tau)\, y(-T\tau+t_0)d\tau, \tag{13}$$

where $p_{k,\mu,-T,n,N,\xi}$ is the associated polynomial.

III. Analysis on the error due to a noise with known statistical properties

Assume now that $y(t_i) = x(t_i) + \varpi(t_i)$ is a noisy measurement of x in a discrete case with an equidistant sampling period T_s, where the noise $\varpi(t_i)$ is a sequence of independent random variables with the same expected value and the same variance. The estimate of the n^{th} order derivative of the signal is given by (13). Since $y(\cdot)$ is a discrete measurement, it needs to use a numerical integration method so as to approximate the integral value in (13).

Let f be a continuous function defined on a bounded interval $f : [0,1] \to \mathbb{R}$. By applying a quadrature formula, the numerical integration approximations of the integral $I = \int_0^1 f(x)\,dx$ are given by:

$$I_m = \sum_{j=0}^{M-1} h \sum_{i=1}^{l} b_i f(x_{(l-1)j} + c_i h), \tag{14}$$

where M and l take values in $\mathbb{N}^*$. As $m = M(l-1)$, we deduce that $h = \frac{1}{M}$ and $x_i = \frac{i}{m}$ for $i = 0,\cdots,m$. The nodes c_i are equal to $c_i = \frac{i-1}{l-1}$ and b_i are the weights of the different classical numerical methods used. For instance, for $l \le 7$, the b_i are given in [19].

A. *Analysis on the error due to a known noise*

By applying a numerical integration to (13), it yields

$$h \sum_{j=0}^{M-1} \sum_{i=1}^{l} b_i\, p_{k,\mu,-T,n,N,\xi}(\tau_{i,j})\, y(t_0 + T\tau_{i,j}),$$

where the $\tau_{i,j} = \tau_{(l-1)j} + c_i h$. In order to simplify the notations, let us denote $p_{k,\mu,-T,n,N,\xi}(\cdot)$ as $p(\cdot)$ and $e_\varpi(t_0;k,\mu,-T,n,N,\xi)$ as $e_\varpi(t_0)$. Consequently, the noisy error contribution $e_{\varpi}(t_0)$ can be given in the discrete case by

$$e_{\varpi,m}(t_0) = h \sum_{j=0}^{M-1} \sum_{i=1}^{l} b_i \, p(\tau_{i,j}) \, \varpi(t_0 + T\tau_{i,j}). \qquad (15)$$

As $e_{\varpi,m}(t_0)$ is a finite sum of independent random variables with the same expected value and the same variance, we can compute the expected value and the variance of $e_{\varpi,m}(t_0)$.

Proposition 3: Let $\varpi(t_i)$ be independent random variables with the same expected value $\bar{\alpha} = E[\varpi]$ and the same variance $\bar{\beta} = var[\varpi]$. The expected value of $e_{\varpi,m}(t_0)$ is given by

$$E[e_{\varpi,m}(t_0)] = \bar{\alpha} \; h \sum_{j=0}^{M-1} \sum_{i=1}^{l} b_i \, p(\tau_{i,j}), \qquad (16)$$

and the variance of $e_{\varpi,m}(t_0)$ is given by

$$var[e_{\varpi,m}(t_0)] = \bar{\beta} \; h^2 \sum_{j=0}^{M-1} \sum_{i=1}^{l} b_i^2 \, p^2(\tau_{i,j}). \qquad (17)$$

Proof. Since $\varpi(t_i)$ are independent with the same expected value and the same variance, the proof can be easily achieved by applying the classical additive property of the expected value function and the variance function. ■

Now we can give two bounds for $e_{\varpi,m}(t_0)$. By using the Bienaymé-Chebyshev inequality, we obtain that for any real number $\gamma > 0$:

$$Pr\left(|e_{\varpi,m}(t_0) - E[e_{\varpi,m}(t_0)]| \geq \gamma \sqrt{Var[e_{\varpi,m}(t_0)]} \right) \leq \frac{1}{\gamma^2}.$$

Then,

$$Pr\left(|e_{\varpi,m}(t_0) - E[e_{\varpi,m}(t_0)]| < \gamma \sqrt{Var[e_{\varpi,m}(t_0)]} \right) > 1 - \frac{1}{\gamma^2},$$

$i.e.$

$$e_{\varpi,m}(t_0) \in \,]M_l \,, M_h[\text{ with a probability } > 1 - \frac{1}{\gamma^2},$$

where $M_l = E[e_{\varpi,m}(t_0)] - \gamma\sqrt{Var[e_{\varpi,m}(t_0)]}$ and $M_h = E[e_{\varpi,m}(t_0)] + \gamma\sqrt{Var[e_{\varpi,m}(t_0)]}$.

B. Analysis on the gaussian error in the estimations

We assume now that the discrete noisy measurement is written as $y(t_i) = x(t_i) + C\varpi(t_i)$, where $C \in \mathbb{R}^+$, and the noise $\varpi(t_i)$ is a sequence of independent random variables with the same standard normal distribution ($\varpi(t_i) \sim \mathcal{N}(0,1)$). So the error $e_{\varpi,m}(t_0)$ is also a gaussian random variable. Since $e_{\varpi,m}(t_0) \sim \mathcal{N}(\hat{\alpha}, \hat{\beta})$ (with $\hat{\alpha} = E[e_{\varpi,m}(t_0)]$ and $\hat{\beta} = var[e_{\varpi,m}(t_0)]$), we have[3]

$$\hat{\alpha} - 2\sqrt{\hat{\beta}} \overset{95.5\%}{\leq} e_{\varpi,m}(t_0) \overset{95.5\%}{\leq} \hat{\alpha} + 2\sqrt{\hat{\beta}}.$$

As the expected value of the noise is equal to zero, then according to Proposition 3, $E[e_{\varpi,m}(t_0)] = 0$. When the parameters k, μ, $-T$, n, N and ξ are chosen, the expression of the polynomial $p(\cdot)$ can be known. We can compute $var[e_{\varpi,m}(t_0)]$ by using Proposition 3, so that we can find out two bounds for the noise error contribution $e_{\varpi,m}(t_0)$:

$$|e_{\varpi,m}(t_0)| \overset{95.5\%}{\leq} M_m(k,\mu,-T,n,N,\xi) \qquad (18)$$

where $M_m(k,\mu,-T,n,N,\xi) = 2\sqrt{var[e_{\varpi,m}(t_0)]}$.

Remark 2: As $\forall t_0 \in I, var[\varpi] = 1$, according to (17) $var[e_{\varpi,m}(t_0)]$ does not depend on t_0, so $M_m(k,\mu,-T,n,N,\xi)$.

IV. EXAMPLE: OBSERVATION OF A NONLINEAR SYSTEM WITH NOISY OUTPUT

Let us consider the following non linear system

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\sin(x_2) + x_1 x_2 + u, \\ y &= x_1. \end{cases} \qquad (19)$$

This system is observable ($x_1 = y$, $x_2 = \dot{y}$ see [12]). Using the above obtained estimators one can reconstruct the state by estimating y_e (obtained with $n = 0$ in formula (13)) which is a filtered estimation of the output, and by estimating its derivative $\dot{y}_e$ (obtained with $n = 1$ in formula (13)). The Figure 1 and Figure 2 show good reconstructions despite the presence of a white noise.

Fig. 1. Noisy output in blue, output without noise in black, and the filtered output using minimal causal estimator in red.

In Figure 2 two time delays appear. They are produced by the corresponding bias term errors, and are predicted by the theory (see in [22], [23]). The obtained estimations taking into account the knowledge of these delays are shown in Figure 3.

Moreover, the noise error contributions in each estimate and the predicted bounds given by (18) are shown in Figure 4, Figure 5 and Figure 6. Let us stress that these bounds are given with a probability of 0.956 which explains why some occurrences are out of these bounds.

[3] $a \overset{c\%}{\leq} b$ means that the probability to have $a \leq b$ is $c\%$.

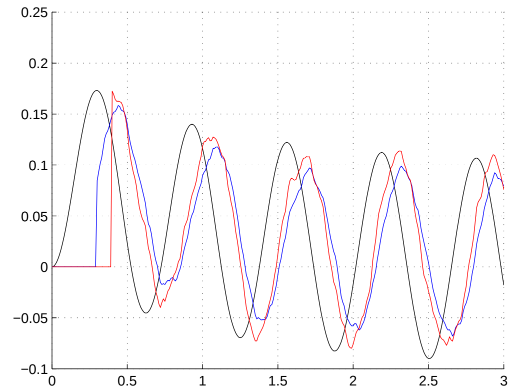

Fig. 2. Output derivative without noise in black, $\dot{y}_e$ using affine causal estimator in red and $\dot{y}_e$ using minimal causal estimator in blue.

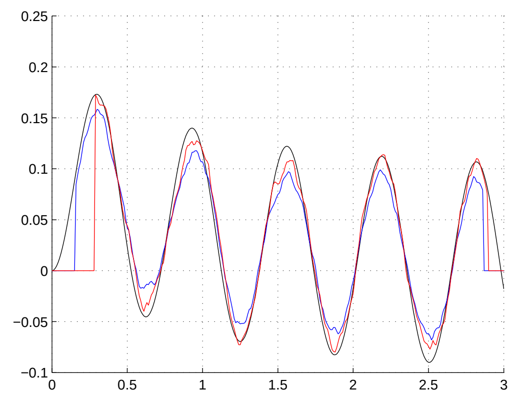

Fig. 3. Output derivative without noise in black, $\dot{y}_e$ using affine causal estimator in red and $\dot{y}_e$ using minimal causal estimator in blue.

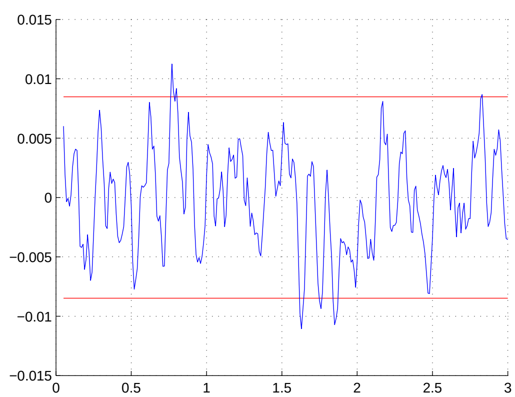

Fig. 4. Noise error contribution in the estimation of x_1 and the predicted bounds given by (18) in red.

Fig. 5. Using a minimal causal estimator: noise error contribution in the estimation of x_2 and the predicted bounds given by (18) in red.

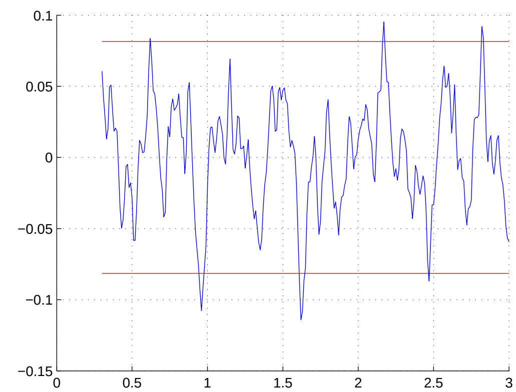

Fig. 6. Using an affine causal estimator: noise error contribution in the estimation of x_2 and the predicted bounds given by (18) in red.

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